

Abstract

A space is n -strong arc connected (n -sac) if for any n points in the space there is an arc in the space visiting them in order. A space is ω -strong arc connected (ω -sac) if it is n -sac for all n . We study these properties in finite graphs, regular continua, and rational continua. There are no 4-sac graphs, but there are 3-sac graphs and graphs which are 2-sac but not 3-sac. For every n there is an n -sac regular continuum, but no regular continuum is ω -sac. There is an ω -sac rational continuum. For graphs we give a simple characterization of those graphs which are 3-sac. It is shown, using ideas from descriptive set theory, that there is no simple characterization of n -sac, or ω -sac, rational continua.

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Strong Arcwise Connectedness

Introduction

In [5] the property of being *n-arc connected* was introduced — a topological space is *n-arc connected* if given any n points in the space, there is an arc in the space containing the points. In this paper we strengthen the condition ‘there is an arc containing the points’ by requiring the arc to traverse the points in a given order. We call this property *n-strong arc connectedness* (abbreviated *n-sac*), and we call a space which is *n-sac* for all n an ω -strongly arc connected space (ω -sac).

Evidently a space is 2-sac if and only if it is arc connected. Many naturally occurring examples of arc connected spaces, especially those of dimension at least two, are ω -sac. For instance it is easy to see that manifolds of dimension at least 3, with or without boundary, and all 2-manifolds without boundary, are ω -sac. But note that the closed disk is 3-sac but not 4-sac (there is no arc connecting the four cardinal points in the order *North*, *South*, *East* and then *West*). We are lead, then, to focus on one-dimensional spaces, and in particular on *curves*: one-dimensional continua (compact, connected metric spaces).

To further hone our focus, we observe that there is a natural obstruction to spaces being 3-sac. Suppose a space X contains an arc-cut point x_1 (in other words, $X \setminus \{x_1\}$ is not arc connected), and fix points x_2 and x_3 for which there is no arc in $X \setminus \{x_1\}$ from x_2 to x_3 . Then no arc in X visits the points x_1, x_2, x_3 in the given order, and thus X is not 3-sac. More generally, see Lemma 1, if removing some $n - 2$ points from a space renders it arc disconnected, then it is not *n-sac*. A continuum is said to be *regular* if it has a base all of whose elements have a finite boundary, and is *rational* if it has a base all of whose elements have a countable boundary. It is well known that all rational continua are curves. From our observation it would seem that

regular curves could only ‘barely’ be n -sac for $n \geq 3$, while rational curves could only ‘barely’ be ω -sac — if, indeed, such spaces exist at all.

This paper investigates the n -sac and ω -sac properties in graphs, regular curves, and rational curves. The paper is divided into five sections, in the first section we formally introduce n -strong arc connectedness, give restrictions on spaces being 4-sac, or more generally n -sac. In particular we show that no planar continuum is 4-sac. In Section 2 we study n -strong arc connectedness in graphs noting that graphs are never 4-sac, and giving a simple (in a precisely defined sense) characterization of those graphs which are 3-sac. In Section 3 we observe that regular curves are never ω -sac, but that there exist, for every n , a regular curve which is n -sac but not $(n + 1)$ -sac. While in Section 4 we construct a locally connected ω -sac rational curve. In contrast to the case with graphs, we have not been able to find a simple characterization of regular n -sac curves. However, in the last section we study the complexity of the set of rational n -strongly arc connected continua as a subset of the space of all subcontinua of \mathbb{R}^N , for $N \geq 3$, and deduce that — *provably* — there is no simple characterization of rational n -sac, or ω -sac, curves. Further, we prove that there is no characterization of n -sac or ω -sac curves (not necessarily rational) less complex than the definition itself. We introduce the machinery from descriptive set theory to make these claims precise, and to prove them, at the start of Section 5. The paper concludes with a discussion of open problems.

1 Preliminaries

In this section we introduce the basic definitions and notation used throughout the paper. Most of the basic notions are taken from [7].

A topological space X is *n -strongly arc connected* (n -sac) if for every distinct x_1, \dots, x_n in X there is an arc $\alpha : [0, 1] \rightarrow X$ and $t_1 < t_2 < \dots < t_n$ from $[0, 1]$ such that $\alpha(t_i) = x_i$ for $i = 1, \dots, n$ — in other words, the arc α ‘visits’ the points in order. Note that we can assume that $t_1 = 0$ and $t_n = 1$, or even that $t_i = (i - 1)/(n - 1)$ for $i = 1, \dots, n$. A topological space is called ω -sac if it is n -sac for every n .

In connection with n -arc connectedness, observe that n -strong arc connectedness implies n -arc connectedness. On the other hand, a simple closed curve is ω -arc connected but is not 4-strongly arc connected, thus the class of n -strongly arc connected spaces is a proper subclass of n -arc connected

spaces.

Lemma 1. *Let X be a topological space. If there is a finite F such that $X \setminus F$ is disconnected, then X is not $(|F| + 2)$ -sac.*

Proof. If F is empty then X is disconnected and hence not 2-sac. So suppose $F = \{x_1, \dots, x_n\}$ for $n \geq 1$. Let U and V be an open partition of $X \setminus F$. Pick x_{n+1} in U and x_{n+2} in V . Consider an arc α in X visiting x_1, \dots, x_n , and then x_{n+1} . Then α ends in U and can not enter V without passing through F . Thus no arc extending α can end at x_{n+2} — and X is not $n + 2$ -sac, as claimed. \square

Corollary 2. *Let X be a topological space.*

- (1) *If there is an open non-dense set U with finite boundary, then X is not $(|\partial U| + 2)$ -sac.*
- (2) *A continuum containing a free arc is not 4-sac.*
- (3) *No compact continuous injective image of an interval is 4-sac.*

Proof. (1) is simply a restatement of Lemma 1. For (2), apply (1) to an open interval inside the free arc. While for (3) note that, by Baire Category, a compact continuous injective image of an interval contains a free arc, so apply (2). \square

Call an arc α in a space X a ‘no exit arc’ if every arc β containing the endpoints of α , and meeting α ’s interior must contain all of α .

Lemma 3. *If a space contains a no exit arc then it is not 4-sac.*

Proof. Let x_1 and x_2 be the endpoints of α . Pick x_3 and x_4 so that x_1, x_3, x_4, x_2 are in order along α . Suppose, for a contradiction, β is an arc visiting the x_i in order. Since x_3 and x_4 are in the interior of α , by hypothesis, β contains α . Now we see that if β enters the interior of α from x_1 then it visits x_3 before x_2 . While if β enters the interior of α from x_2 it visits x_4 before x_3 . Either case leads to a contradiction. \square

Proposition 4. *No planar continuum is 4-sac.*

Proof. Let K be a plane continuum. If it is not arc connected then it is not 2-sac, so suppose K is arc connected. Pick \mathbf{x}_- (respectively, \mathbf{x}_+) in K to have minimal x -coordinate (resp., maximal x -coordinate). If \mathbf{x}_- and \mathbf{x}_+ have the same x -coordinate, then X is an arc, and so not 3-sac.

Otherwise, translating the mid point between \mathbf{x}_- and \mathbf{x}_+ to the origin, shearing in the y -coordinate only to move \mathbf{x}_- and \mathbf{x}_+ onto the x -axis, and then scaling, we can assume without loss of generality that $\mathbf{x}_- = (-1, 0)$, $\mathbf{x}_+ = (+1, 0)$ and $K \subseteq [-1, 1] \times \mathbb{R}$.

There is an arc α in K from \mathbf{x}_- to \mathbf{x}_+ . Some subarc, α' , of α meets $\{-1\} \times \mathbb{R}$ and $\{+1\} \times \mathbb{R}$ in just one point (each). If for every x in the interval $(-1, 1)$ the vertical line $\{x\} \times \mathbb{R}$ meets K in just one point, then α' is a free arc, and K is not 4-sac, as claimed.

Otherwise there is an $x_0 \in (-1, 1)$ such that there are two distinct points \mathbf{x}_3 and \mathbf{x}_4 in $K \cap (\{x_0\} \times \mathbb{R})$. We can suppose \mathbf{x}_3 has minimal y -coordinate, y_3 , while \mathbf{x}_4 has maximal y -coordinate, y_4 . Assume, for a contradiction, that there is an arc β from $\mathbf{x}_1 = \mathbf{x}_-$ to \mathbf{x}_4 visiting $\mathbf{x}_2 = \mathbf{x}_+$ and \mathbf{x}_3 in order. Let β_1 be the subarc of β from \mathbf{x}_1 to \mathbf{x}_2 and β_3 be the subarc of β from \mathbf{x}_3 to \mathbf{x}_4 . Note that $\beta_1 \cap \beta_3 = \emptyset$, and so β_1 meets $\{x_0\} \times \mathbb{R}$ only inside $\{x_0\} \times (y_3, y_4)$. Hence the line $L = (-\infty, -1) \times \{0\} \cup \beta_1 \cup (+1, +\infty) \times \{0\}$ splits the plane into two disjoint open sets, U_3 containing \mathbf{x}_3 , and U_4 containing \mathbf{x}_4 . However β_3 is supposed, on the one hand, to be an arc from \mathbf{x}_3 to \mathbf{x}_4 , and so must cross L , and on the other hand, is forced to be disjoint from each part of L : β_1 (by choice of β) and both $(-\infty, -1) \times \{0\}$ and $(+1, +\infty) \times \{0\}$ (since $K \subseteq [-1, 1] \times \mathbb{R}$) — contradiction. \square

2 Graphs

From Corollary 2 (2) it is immediate that no graph is 4-sac. Since only connected graphs will be considered, all graphs are 2-sac. In this section we give a characterization of 3-sac graphs. In fact, we show that for a general continuum X the property of being 3-sac is equivalent to the intensively studied property of being cyclicly connected (any two points in X lie on a circle).

We begin this section by noticing that the triod and the figure eight continuum are not 3-sac, while the circle and the theta curve continuum are 3-sac. In [2, Theorem 1], Bellamy and Lum proved:

Theorem 5. *For a continuum X , the following are equivalent:*

- (1) *X is cyclicly connected;*
- (2) *X is arc connected, has no arc-cut point, and has no arc end points.*

Using the previous theorem we obtain the following characterization of 3-sac continua.

Proposition 6. *For a continuum X , the following are equivalent:*

- (1) *X is cyclicly connected;*
- (2) *X is 3-sac;*
- (3) *Any three points in X lie either on a circle or on a theta curve.*

Proof. (3) \Rightarrow (2): this follows from the fact that the circle and theta curve are both 3-sac.

(2) \Rightarrow (1): if X is 3-sac, then for any $x \in X$, there is an arc that contains x in its interior. So X has no endpoints. If X has an arc-cut point then, by Lemma 1, X is not 3-sac. Now by Theorem 5 X is cyclicly connected.

(1) \Rightarrow (3): Let $x, y, z \in X$. By (1) there is a circle C in X containing x and y . If $z \in C$ we are done. If $z \notin C$, then by (1) there are two arcs, α and β , from z to x , that only meet at the endpoints. Let a and b be the points when α and β first intersect C and let α' and β' be parts of α and β from z to a and b respectively. If $a \neq b$, $C \cup \alpha' \cup \beta'$ is the desired theta curve. If $a = b$, then a is not an arc-cut point by the above theorem, so there is an arc γ from z to some point on C other than a that misses a . Let γ' be part of γ that starts in $(\alpha' \cup \beta') - \{a\}$, ends in $C - \{a\}$ and does not meet $C \cup \alpha' \cup \beta'$ otherwise. Then $C \cup \alpha' \cup \beta' \cup \gamma'$ contains a theta curve that passes through x, y, z . \square

Notice that for finite graphs cyclicly connected is equivalent to having no cut points.

Corollary 7. *For a finite graph X , the following are equivalent:*

- (1) *X has no cut points;*
- (2) *X is 3-sac;*
- (3) *Any three points in X lie either on a circle or on a theta curve.*

It follows immediately from this characterization of 3-sac graphs, combined with Lemma 3.2 and the proof of Proposition 3.4 of [5] that the set of

3-sac graphs, considered as a subspace of $\mathcal{C}(I^3)$, the hyperspace of all subcontinua of the cube, is the intersection of a $G_{\delta\sigma}$ set (countable union of countable intersections of open sets) and a $F_{\sigma\delta}$ set (countable intersection of countable unions of closed sets), but is neither a $G_{\delta\sigma}$ set nor a $F_{\sigma\delta}$ set.

An alternative way of expressing this fact, is that there is a characterization of 3-sac graphs of the logical form $\forall p \exists q \forall r \theta(p, q, r) \wedge \exists p \forall q \exists q \phi(p, q, r)$, where the quantifiers run over *countable* sets, and $\theta(p, q, r)$, $\phi(p, q, r)$ are simple (boolean) sentences. However no logically simpler description of the 3-sac graphs exists. This should be contrasted with the fact that the *definition* of 3-sac — ‘ $\forall x_1, x_2, x_3 \in X \exists \text{ arc } \alpha \dots$ ’ — the two quantifiers run over *uncountable* sets. Thus the given characterization of 3-sac graphs is significantly simpler than the definition.

3 Regular Curves

In this section we construct, for every $n \geq 3$, an n -sac regular continuum, then using the Finite Gluing Lemma (Lemma 9) we show that for any $n \geq 2$ there is a regular continuum — therefore rational — that is n -sac but not $(n + 1)$ -sac.

We start the section by introducing the basic elements needed to construct an n -sac regular continuum.

Fix $N \geq 3$. Suppose v_1, \dots, v_k are affinely independent points in \mathbb{R}^{N-1} . Denote by $\langle v_1, \dots, v_k \rangle$ the convex span of v_1 through v_k . Then $\langle v_1, \dots, v_k \rangle$ is a k -simplex. We call the points v_1, \dots, v_k the vertices of $\langle v_1, \dots, v_k \rangle$. For any $i \neq j$, we call $\langle v_i, v_j \rangle$ the edge from v_i to v_j , and we let $v_i \wedge v_j$ be the midpoint between v_i and v_j . Note that the space of all edges, $\bigcup_{i < j \leq k} \langle v_i, v_j \rangle$, of $\langle v_1, \dots, v_k \rangle$, is a complete graph on the vertices v_1, \dots, v_k .

Fix v_1, \dots, v_N affinely independent points in \mathbb{R}^{N-1} , for example let v_1 through v_{N-1} be the standard unit coordinate vectors, and $v_N = \mathbf{0}$. Define the operation *Trix* taking a simplex $\langle v_1, \dots, v_N \rangle$ and returning a set of simplices, $\{\langle v_i, v_i \wedge v_j : i \neq j \rangle : i = 1, \dots, N\}$. Inductively define sets of N -simplices as follows: $\mathcal{T}_0^N = \{\langle v_1, \dots, v_N \rangle\}$, and $\mathcal{T}_{m+1}^N = \bigcup_{S \in \mathcal{T}_m^N} \text{Trix}(S)$. Let $T_m^N = \bigcup \mathcal{T}_m^N$, and $T^N = \bigcap_m T_m^N$. Then T^N is a regular continuum we call the N -trix. Observe that the 3-trix is the Sierpinski triangle, and the 4-trix is the tetrax (hence our name for these continua).

Some additional notation. Given a simplex $S = \langle v_1, \dots, v_N \rangle$, let $\mathcal{T}_1 = \text{Trix}(S)$, and $T_1 = \bigcup \mathcal{T}_1$. Take any element of \mathcal{T}_1 , say $S_i = \langle v_i, v_i \wedge v_j : j \neq i \rangle$.

Call the point v_i the external vertex of S_i , and call the points $v_i \wedge v_j$, for $j \neq i$, the internal vertices of S_i . For any S in \mathcal{T}_1 , denote the external vertex of S by $v(S)$. For any two elements, S and S' of \mathcal{T}_1 , denote the (unique) internal vertex common to S and S' by $S \wedge S'$. Note that $\{S \wedge S'\} = S \cap S'$. Further for any x in T_1 , fix an element, $S(x)$, of \mathcal{T}_1 containing x .

It is easy to verify directly, or by applying Theorem 5 and Proposition 6 that all N -trixes are 3-sac. From Lemma 4 we see that the 3-trix (i.e. the Sierpinski triangle) is not 4-sac. Rather unexpectedly, the 4-trix (i.e. the tetrix) is also not 4-sac. To see this consider the sequence of points $x_1 = v_1 \wedge v_2, x_2 = v_3, x_3 = v_2, x_4 = v_1 \wedge v_3$. Using computer calculations, we have verified that the 5-trix is 5-sac but not 6-sac. It is not clear to the authors for which n a given N -trix is n -sac, but we can show that there is no upper bound on the natural numbers, n , that can be realized.

Lemma 8. *Fix $n \geq 3$. Let $N = 6n^2 + 12n + 1$. The N -trix is n -sac.*

Proof. Let $T = T^N$, the N -trix, and — since N is fixed to be $6n^2 + 12n + 1$ — otherwise suppress the superscript N . Take any n points in T , say x_1, \dots, x_n . Then there is a minimum $m \geq 1$ such that the x_i are in distinct simplices in \mathcal{T}_m . Further there is a maximum m' so that all the x_i are in the some simplex S of $\mathcal{T}_{m'}$. If there is an arc in $S \cap T$ visiting the points in order, then that same arc visits the points in order inside T . So without loss of generality, we can suppose that $m' = 0$, $S = T_0$, and the points x_1, \dots, x_n (obviously) each lie in an element of \mathcal{T}_1 , but not all in the same element. Now call m the height of the points, x_1, \dots, x_n .

There are $N = 6n^2 + 12n + 1$ elements of \mathcal{T}_1 . Each of the n points, x_i , can only be in at most 2 members of \mathcal{T}_1 . Hence we can find a subset \mathcal{E} of \mathcal{T}_1 such that \mathcal{E} has at least $3n + 1$ members, and no point x_i is in any element of \mathcal{E} . The lemma now follows from the next claim, which we prove by induction on m .

Claim: for each $m \geq 0$, points x_1, \dots, x_n of height m , and subset \mathcal{E} of \mathcal{T}_1 , such that $|\mathcal{E}| > 3n$ and $\mathcal{E} \cap \{S(x_i) : i \leq n\} = \emptyset$ (for any choice of $S(x_i)$), there is an arc α visiting the points x_1, \dots, x_n in order, and, for $i = 1, \dots, n$, disjoint arcs (called ‘spurs’), β_i from x_i to the external vertex, $v(E)$, of some E in \mathcal{E} .

Base Step, $m = 1$. Since $m = 1$, we can assume that the sets $S(x_i)$, for $i = 1, \dots, n$, are distinct.

Pick some C in \mathcal{E} . Let $\mathcal{E}' = \mathcal{E} \setminus \{C\}$. For each $i \leq n$, and $j = 1, 2, 3$, pick distinct $E_{i,j}$ from \mathcal{E}' . For each $i \leq n$, pick three disjoint arcs in $S(x_i)$: α_i^- from x_i to $S(x_i) \wedge E_{i,1}$, α_i^+ from $S(x_i) \wedge E_{i,2}$ to x_i , and β_i from x_i to $S(x_i) \wedge E_{i,3}$. Extend β_i by following the edge in $E_{i,3}$ to $v(E_{i,3})$. These β_i are the required ‘spurs’. Denote by Ω the set of simplices, $E_{i,3}$, containing these spurs.

For $i < n$, let α_i be the arc formed by following the natural edges (of elements of \mathcal{T}_1) between these vertices in the prescribed order: $S(x_i) \wedge S(E_{i,1})$, $S(E_{i,1}) \wedge S(C)$, $S(C) \wedge S(E_{i+1,2})$ and $S(E_{i+1,2}) \wedge S(x_{i+1})$. Let α be the path obtained by following these arcs in the given order: $\alpha_1^-, \alpha_i, \alpha_2^+, \alpha_2^-, \alpha_2, \alpha_3^+, \dots, \alpha_i^-, \alpha_i, \alpha_{i+1}^+, \dots$, and finally $\alpha_{n-1}^-, \alpha_{n-1}, \alpha_n^+$. Since all the vertices appearing in the definition of the α_i s are distinct, α is a path which does not cross itself, and so is an arc, which, by construction, visits the points x_1, \dots, x_n in order.

Inductive Step. We assume the claim is true when the points come from a level $< m$. Prove for points on level m . First observe that for any S in \mathcal{T}_1 , $S \cap \mathcal{T}_m$ is homeomorphic to \mathcal{T}_{m-1} .

For clarity we will use I_l to denote the set $\{1, 2, \dots, l\}$. Let x_1, x_2, \dots, x_n be n points in T of height m and $\{S^1, S^2, \dots, S^k\} = \{S(x_1), \dots, S(x_n)\}$. For each $i \in I_k$ let $x_{(i,1)}, x_{(i,2)}, \dots, x_{(i,k_i)}$ be a reenumeration of all the x_j s in S^i such that if $x_t = x_{(i,s)}$ and $x_l = x_{(i,r)}$ then $(i, s) < (i, r)$ if and only if $t < l$. For all S in \mathcal{T}_1 , let \mathcal{S}_{m-1} denote $S \cap \mathcal{T}_m$. In each S^i pick $6n + 1$ -many simplices of \mathcal{S}_1^i that do not contain any of $x_{(i,j)}$ s, none of them share external vertices, and none contain the external vertex of S^i ; this can be done since \mathcal{S}_1^i consists of $6n^2 + 13n + 1$ simplices and S^i contains at most $n - 1$ elements of $\{x_1, x_2, \dots, x_n\}$. Let this set of simplices be \mathcal{E}_i .

In the next step we will choose the simplices that will allow us to construct an arc between two consecutive x_i s, whenever they lie on different elements of \mathcal{T}_1 .

For each $i \in I_k$ let Υ_i be a set of simplices $Y_{(i,j)}$ in \mathcal{T}_1 given as follows:

1. For $j < k_i$,
 - (a) If $x_{(i,j)}$ and $x_{(i,j+1)}$ are not consecutive points in $\{x_1, x_2, \dots, x_n\}$, then pick $Y_{(i,j)}$ such that

$$Y_{(i,j)} \notin \{S^1, S^2, \dots, S^k\} \cup \left\{ \bigcup_{t=1}^{i-1} \Upsilon_t \right\} \cup \{Y_{(i,1)}, Y_{(i,2)}, \dots, Y_{(i,j-1)}\},$$

$Y_{(i,j)} \cap \{x_1, x_2, \dots, x_n\} = \emptyset$, and such that $Y_{(i,j)} \wedge S^i$ lies in an element of \mathcal{S}_{m-1} different from the elements containing the $x_{(i,j)}$ s and the elements of \mathcal{E}_i .

- (b) If $x_{(i,j)}$ and $x_{(i,j+1)}$ are consecutive points in $\{x_1, x_2, \dots, x_n\}$, then do nothing.

2. For $j = k_i$,

- (a) If $x_{(i,k_i)} \neq x_n$, then pick $Y_{(i,k_i)}$ as above, satisfying the conditions on (a).
- (b) If $x_{(i,k_i)} = x_n$, then do nothing.

3. For $j = 1$ (in some cases we are selecting twice for $j = 1$),

- (a) If $x_{(i,1)} \neq x_1$, then pick $Y_{(1,0)}$ as in (1), satisfying the conditions on (a).
- (b) If $x_{(i,1)} = x_1$, then do nothing.

Denote by $y_{(i,j)}$ the vertex $S^i \wedge Y_{(i,j)}$.

Now, in each sequence $\{x_{(i,1)}, x_{(i,2)}, \dots, x_{(i,k_i)}\}$ insert the points $y_{(i,j)}$ (if they exist) as follows: $y_{(i,0)}$ before $x_{(i,1)}$, and $y_{(i,j)}$ immediately after corresponding $x_{(i,j)}$. So, for each $i \in I_k$, we have constructed a sequence l_i in S^i such that a point $y_{(i,j)}$ lies between $x_{(i,j)}$ and $x_{(i,j+1)}$ only if $x_{(i,j)}$ and $x_{(i,j+1)}$ are not consecutive points on $\{x_1, x_2, \dots, x_n\}$. Observe that for each $i \in I_k$, the set of points $x_{(i,1)}, x_{(i,2)}, \dots, x_{(i,k_i)}$ have, in S^i , height at most $m-1$, hence, by the choice of ys , the sequence of points l_i also has height at most $m-1$.

By Inductive Hypothesis, applied to S^i , l_i and \mathcal{E}_i , there is an arc α_i in S^i visiting the points of l_i in order, and disjoint spurs β_a for each $a \in l_i$ to external vertices of some elements in \mathcal{E}_i .

Construction of an arc through x_1, x_2, \dots, x_n : Pick $C \in \mathcal{E}$ containing none of the ys or xs . For each $i \in I_{n-1}$, let γ_i be the arc connecting x_i to x_{i+1} given as follows: If x_i and x_{i+1} are in the same S^t then γ_i is the subarc of α_t connecting them. If not, then $x_i = x_{(p,j)}$, $x_{i+1} = x_{(r,l)}$ and $y_{(p,j)}, y_{(r,l-1)}$ exist. Let $\gamma_i = \gamma_i^1 \cup \gamma_i^2 \cup \gamma_i^3 \cup \gamma_i^4 \cup \gamma_i^5$, where $\gamma_i^1, \gamma_i^2, \gamma_i^3, \gamma_i^4, \gamma_i^5$ are as follows:

1. γ_i^1 is the subarc of α_p from $x_{(p,j)}$ to $y_{(p,j)}$ if possible or else a spur from $x_{(p,j)}$ to some vertex u of S^p , whichever is unused yet. In any case, there is a simplex U in \mathcal{T}_1 such that $u = S^p \wedge U$ or $y_{(p,j)} = S^p \wedge U$. Let γ_i^2 be the edge in U connecting $S^p \wedge U$ to $U \wedge C$.

2. similarly as in (1), γ_i^3 is the subarc α_r from $x_{(r,l)}$ to $y_{(r,l-1)}$ if possible or else a spur from $x_{(r,l)}$ to some vertex v of S^r , whichever is unused yet. In any case, there is a simplex V in \mathcal{T}_1 such that $v = S^r \wedge V$ or $y_{(r,l-1)} = S^r \wedge V$. Let γ_i^4 be the edge in V connecting $S^r \wedge V$ to $V \wedge C$.
3. let γ_i^5 be the edge in C connecting $U \wedge C$ and $V \wedge C$.

Let $\alpha = \bigcup_{i=1}^{n-1} \gamma_i$. Because of how y s and spur destinations were picked, α is an arc that visits the points x_1, x_2, \dots, x_n in order.

Construction of spurs to external vertices of elements of \mathcal{E} : suppose we have constructed spurs for all x_l , $l < i$ and $x_i \in S^j$. If spur β_{x_i} of x_i in S^j is not contained in α then it only intersects it at x_i , extend β_{x_i} as follows: let v_i be the other endpoint of β_{x_i} . Let V_i be the simplex in $\mathcal{T}_1 \setminus (\{\bigcup_{i=1}^k \Upsilon_i\} \cup (\bigcup_{i=1}^k S^i) \cup C)$ that intersects S^j at v_i . Pick any simplex E_i in $\mathcal{E} \setminus C$ that has not been picked for previous spur constructions. The spur β_i consist of β_{x_i} , followed by the edge in V_i connecting v_i and $E_i \wedge V_i$, and the edge in E_i connecting $E_i \wedge V_i$ with $v(E_i)$.

Suppose β_{x_i} is contained in α . Observe that in this case $x_i = x_{(j,r)}$ and $x_{(j,r-1)}$ are not consecutive points of $\{x_1, x_2, \dots, x_n\}$, otherwise β_{x_i} would not be contained in α . Hence $y_{(j,r-1)}$ exists. Let δ be the subarc of α_j connecting $x_{(j,r)}$ to $y_{(j,r-1)}$, let γ be the subarc of α connecting $x_{(j,r-1)}$ to $y_{(j,r-1)}$, and let ω be the other end point of the spur $\beta_{y_{(j,r-1)}}$. By construction, $\alpha \cap \delta = \{x_{(j,r)}, y_{(j,r-1)}\}$ and $\alpha \cap \beta_{y_{(j,r-1)}} = \{y_{(j,r-1)}\}$.

Since the diameters of the simplices in \mathcal{T}_t approach zero as t increases, there exists, for a sufficiently large t , a simplex Λ in \mathcal{S}_t^j with the following properties:

1. $y_{(j,r-1)}$ is a vertex of Λ ,
2. $x_{(j,r)}$, $x_{(j,r-1)}$, $\omega \notin \Lambda$,
3. $\Lambda \cap \alpha$ is connected, and
4. Λ does not intersect any spur, except for $\beta_{y_{(j,r-1)}}$.

By the choice of Λ , the arcs δ , γ and $\beta_{y_{(j,r-1)}}$ intersect Λ at different vertices of Λ , say a , b , c respectively. Then revise α to go from b to $y_{(j,r-1)}$ through an edge of Λ and let β consist of the following parts: the subarc of δ from $x_i = x_{(j,r)}$ to a , followed by the edge in Λ from a to c , and followed by the subarc of $\beta_{y_{(j,r-1)}}$ from c to ω . Now extend β as in the previous case to get the spur β_i for x_i . \square

Lemma 9 (Finite Gluing). *If X and Y are $(2n - 1)$ -sac, and Z is obtained from X and Y by identifying pairwise $n - 1$ different points of X and Y , then Z is n -sac but not $(n + 1)$ -sac.*

Proof. Let z_1, z_2, \dots, z_n be any n points in Z . For each i , if $z_i \in X \setminus Y$ and $z_{i+1} \in Y \setminus X$ or $z_i \in Y \setminus X$ and $z_{i+1} \in X \setminus Y$, pick $z_{(i,i+1)} \in (X \cap Y) \setminus \{z_1, z_2, \dots, z_n, z_{(1,2)}, z_{(2,3)}, \dots, z_{(i-1,i)}\}$ (if $z_{(1,2)}, z_{(2,3)}, \dots, z_{(i-1,i)}$ were picked). This is possible since $|X \cap Y| = n - 1$. Let \mathcal{Z} be the sequence of z_j s with $z_{(i,i+1)}$ s inserted between z_i and z_{i+1} whenever they exist. And let \mathcal{Z}_X be the sequence derived from \mathcal{Z} by deleting the terms that do not belong to X . Define \mathcal{Z}_Y similarly. Since elements of \mathcal{Z}_X come either from $\{z_1, z_2, \dots, z_n\}$ or from $X \cap Y$, $|\mathcal{Z}_X| \leq 2n - 1$. Similarly, $|\mathcal{Z}_Y| \leq 2n - 1$. Let β be an arc in X going through elements of \mathcal{Z}_X in order and γ be an arc in Y going through elements of \mathcal{Z}_Y in order. Let a_1, a_2, \dots, a_k be $z_{(1,2)}, z_{(2,3)}, \dots, z_{(n-1,n)}$ whenever they exist, respectively. Without loss of generality, suppose $z_1 \in X$. Define α to be the union of the following arcs:

1. the subarc of β from z_1 to a_1 ;
2. the subarc of γ from a_1 to a_2 ;
3. the subarc of β from a_2 to $a_3 \dots$
4. the subarc of β or γ (depending on whether k is even or odd) from a_k to z_n .

Note that α is an arc visiting the points z_1, z_2, \dots, z_n in order. Hence Z is n -sac. The fact that Z is not $(n + 1)$ -sac follows by Lemma 1. \square

Observe that for $n \geq 2$, Lemma 8 implies the existence of a $(2n - 1)$ -sac regular continuum G . Hence by Lemma 9 applied to two disjoint copies of G , there exists an n -sac regular continuum that is not $(n + 1)$ -sac. We summarize this in the following theorem.

Theorem 10. *For every $n \geq 2$ there exists a n -sac regular continuum that is not $(n + 1)$ -sac.*

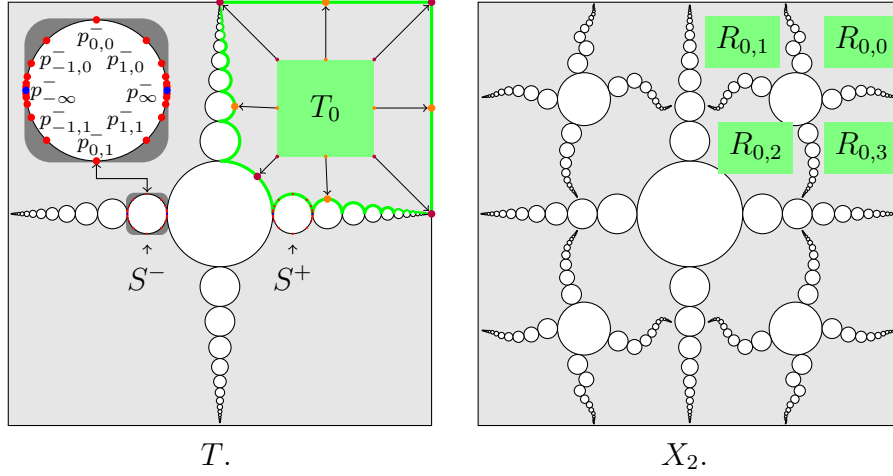
4 Rational Curves

In this section we construct an ω -sac rational continuum. The motivating idea for the construction is as follows. Recall that the closed disk is 3-sac but not 4-sac, and that this is because any arc visiting the cardinal points *North*, *South*, and then *East* (while avoiding *West*) cuts the disk in two, and can not be extended on to *West*. Evidently if we add a handle to the disk, we can use this handle as a bridge from *East* to *West*, and this new space is 4-sac. (In fact a closed disk with a handle is 6-sac but not 7-sac.) If we add more and more handles then the derived space will be n -sac for larger and larger n . So to construct our example of an ω -sac rational continuum we start by modifying Charatonik's description of an example due to Urysohn. This is a rational continuum such that removing no finite subset disconnects it — and so does not fail to be ω -sac for that reason. However, like the closed disk, it is not 4-sac for another reason: it is planar. So we further modify the space by adding an infinite and dense set of handles. This has been done so as to preserve rationality.

Theorem 11. *There is a locally connected rational continuum which is ω -sac.*

Proof. Write $B(\mathbf{x}, r)$ for the open disk in the plane of radius r centered at \mathbf{x} . Write $S(\mathbf{x}, r)$ for the boundary circle of $B(\mathbf{x}, r)$. Pick any monotone sequence $(x_n)_{n=0}^\infty$ in $(0, 1)$ increasing to 1. Let $c_0 = 0$, $r_0 = x_0$ and $c_n = (x_n + x_{n-1})/2$, $r_n = (x_n - x_{n-1})/2$ for $n \geq 1$. Let θ be rotation of the plane by 90° clockwise. Let $U = \bigcup_{i=0}^3 \theta^i (\bigcup_{n=0}^\infty B((c_n, 0), r_n))$.

Let $T = [-1, +1]^2 \setminus U$. Let S be the geometric boundary of T , so $S = \bigcup_{i=0}^3 \theta^i (\bigcup_{n=0}^\infty S((c_n, 0), r_n)) \cup \partial[-1, 1]^2$. Let $S^- = S((-c_1, 0), r_1)$ and $S^+ = S((c_1, 0), r_1)$ (the two circles in S immediately to the left and right of the center circle). For $i=0$ (respectively, $i=1$) pick a two sided sequence of points, $(p_{m,i}^-)_{m \in \mathbb{Z} \cup \{\pm\infty\}}$, on the top (respectively, bottom) edge of S^- converging on the left to $p_{-\infty,i}^- = (-x_1 - r_1, 0)$ (the leftmost point of S^-) and on the right to $p_{\infty,i}^- = (-x_1 + r_1, 0)$ (the rightmost point of S^-). Find a corresponding pair, $(p_{m,i}^+)_{m \in \mathbb{Z} \cup \{\pm\infty\}}$ for $i = 0$, and 1 of double sequences on the top and bottom edges of S^+ converging to the leftmost and rightmost points of S^+ . Let $T_i = \theta^i ([0, 1]^2 \cap T)$ for $i = 0, 1, 2, 3$. Note that each T_i is a topological rectangle with natural 'corners' and 'midpoints' of the sides.



Let $X_1 = T$, $R_{(i)} = T_i$ and $S_1 = S$. Let $h_{(0)}$ be a homeomorphism of $[-1, +1]^2$ with T_0 carrying top-right corner to top-right corner etc, and midpoints to midpoints. Let $h_{(i)} = \theta^i \circ h_{(0)}$ for $i = 1, 2, 3$.

Inductively, suppose we have continuum X_n , geometric boundary S_n , and for each $\sigma \in \Sigma_n = \{0, 1, 2, 3\}^n$ a rectangle R_σ and a homeomorphism h_σ of $[-1, +1]^2$ with R_σ . Fix a σ for a moment. Then R_σ has four subrectangles $h_\sigma(T_i)$. For $i = 0, 1, 2, 3$ let $h_{\sigma \frown i}$ be a homeomorphism of $[-1, +1]^2$ with $h_\sigma(T_i)$ taking corners to corners etcetera. Let $R_{\sigma \frown i} = h_\sigma(T_i) = R_\sigma \setminus h_{\sigma \frown i}(U)$. Let $X_{n+1} = \bigcup_{\sigma \in \Sigma_n} \bigcup_{i=0}^3 R_{\sigma \frown i} = \bigcup_{\sigma \in \Sigma_{n+1}} R_\sigma$. Let S_{n+1} be the natural geometric boundary. Note that $X_{n+1} \subseteq X_n$.

Let $X = \bigcap_n X_n$. Then X is a variant of Charatonik's description of Urysohn's locally connected, rational continuum in which every point has countably infinite order, see [4]. Thus X is rational (and locally connected). Since it is planar it is not 4-sac. Note that each $R_\sigma \cap X$ has a countable boundary contained in the sides of R_σ . Call a side of R_σ *finite* if it contains only finitely many boundary points. A side containing infinitely many boundary points, is said to be *infinite*.

For each n and σ in Σ_n , there are two circles, $h_\sigma(S^-)$ and $h_\sigma(S^+)$. Identify, for all $m \in \mathbb{Z} \cup \{\pm\infty\}$ and $i \in \{0, 1\}$, the points $h_\sigma(p_{-m,i}^-)$ and $h_\sigma(p_{m,i}^+)$ (creating a 'rational bridge' between the circles). Note that the diameters of the circles shrink to zero with n . It follows that the resulting quotient space, Y , is a locally connected, rational continuum. We show that Y is ω -sac.

Fix distinct points x_1, \dots, x_n in Y . The diameters of the rectangles, R_σ for $\sigma \in \Sigma_m$, shrink to zero with m , so we can find an N such that if $i \neq j$, $x_i \in R_\sigma$, and $x_j \in R_\tau$ where $\sigma, \tau \in \Sigma_N$ then R_σ and R_τ are disjoint. For each

i , let R_i be the unique R_σ containing x_i .

Subdivide the square $r = (-1, +1)^2$ into four subsquares $r_{(i)} = \theta^i([0, 1]^2) \cap r$. And continue subdividing to get a final subdivision of $(-1, +1)^2$ into subsquares r_σ for $\sigma \in \Sigma_N$. Note that two squares r_σ and r_τ are adjacent if and only if the corresponding rectangles R_σ and R_τ are adjacent. For each σ in Σ_N , consider R_σ . It has four sides, at most two are finite sides. For each finite side remove the line segment in r which is the corresponding side in r_σ . The result r' is an open, connected subset of the plane. It follows that r' is ω -sac. Hence there is an arc α' which visits the interior of the squares r_i in order: r_1, r_2, \dots, r_n (indeed we can suppose α' visits the centers of the r_i in turn). Further, we can suppose that α' consists of a finite union of horizontal or vertical line segments of the form $\{p/q\} \times J$ or $J \times \{p/q\}$ where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and J is a closed interval. Let M be a common denominator of all the denominators (qs) used. Then α' is an arc on the grid $r' \cap \left(\left(\bigcup_{p \in \mathbb{Z}} \{p/M\} \times \mathbb{R} \right) \cup \left(\bigcup_{p \in \mathbb{Z}} \mathbb{R} \times \{p/M\} \right) \right)$.

Consider X_1 . There is a connected chain of circles, V_0 , in X_1 from the bottom edge to the top, and a connected chain of circles, H_0 , from the left side to the right. Note that V_0 and H_0 are in X . Now consider X_2 . There is a connected chain of circles to the right of V_0 from the top edge to a circle in H_0 , and another to the right of V_0 from the bottom edge to a circle in H_0 . By construction, both chains end at the *same circle* of H_0 . Call the union of these two chains, along with the circle they connect to, V_1 . It is a vertical connected chain of circles from the top edge to the bottom. Similarly, there is a vertical connected chain of circles, V_{-1} from the top edge to the bottom, lying to the left of V_0 . Further there are two horizontal connected chains of circles, H_1 and H_{-1} , above and, respectively, below, H_0 . Observe that V_0 and $V_{\pm 1}$ are disjoint, as are H_0 and $H_{\pm 1}$. Together these six chains form a three-by-three ‘grid’ in X . Repeating, we find a ‘grid’ of horizontal, $H_{\pm n}$ and vertical, $V_{\pm n}$, chains, all in X , where the $H_{\pm n}$ converges to the left and right sides, $\{\pm 1\} \times [-1, 1]$ and $V_{\pm n}$ converges to the top and bottom edges $[-1, 1] \times \{\pm 1\}$.

Now consider a rectangle R_σ for some σ in Σ_N . It has at least two ‘infinite’ sides. For concreteness let us suppose that the bottom and right sides of R_σ are infinite, with the limit point on the bottom edge being to the right, and the limit along the right edge being at the top (all other cases are very similar). The vertical chains, V_n in X , for $n \in \mathbb{N}$, have analogues in R_σ . By construction, each V_n meets the bottom edge of R_σ in an arc of a

circle whose ends are points in the R_τ ‘below’ R_σ . Extend V_n to include this arc. Repeat at the top edge, if it is infinite. Apply the same procedure to the horizontal chains, H_n for $n \in \mathbb{N}$. The horizontal chains, and respectively the vertical chains, remain disjoint. Note that, by construction, if R_τ is the rectangle ‘below’ R_σ , then the n th vertical chain in R_σ connects to the n th vertical chain in R_τ (and similarly for the rectangle to the right of R_σ). If x_i is in R_σ but not on the geometric boundary of R_σ , then let P_i be sufficiently large that x_i is to the left of V_{P_i} and below H_{P_i} .

Now let P be the maximum of the P_i . Return to an individual rectangle, R_σ , as in the previous paragraph. Take the union of the vertical chains, V_P, \dots, V_{P+M} , and the horizontal chains, H_P, \dots, H_{P+M} . Take the union now over all σ in Σ_N . This gives a ‘grid’, G , naturally containing an isomorphic copy of the grid G' in r' . Think of the grid, G , as a graph, and α' as an edge arc in this graph. Then we can realize the arc α' in G' as a connected chain of circles in X . Evidently (by traveling along the ‘top’ or ‘bottom’ edges of the circles in the union) we can extract an arc, α_* , contained in this union. The arc α_* visits the R_i in order. Note that α_* is not (necessarily) an arc in Y , but it can easily be modified to be so, call this arc, α_0 .

To complete the proof, we modify α_0 , to another arc α in Y , which visits the points x_i in order. As α_0 visits the R_i in turn, there are subarcs β_i of α_0 , where β_i comes before β_j if $i < j$, such that β_i crosses from one infinite edge of R_i to another (along the ‘grid’ G inside R_i). We will replace β_i in α_0 by another subarc, visiting x_i , contained inside R_i , with the same start and end as β_i , but otherwise disjoint from the ‘grid’ G . Doing this for all i , gives the arc α in Y .

Fix i . Again for concreteness, orient $R_i = R_\sigma$ as above. Suppose that β_i enters R_i at y_i , a point on the bottom edge, and exits at z_i , a point on the right edge. Pick Q in \mathbb{N} sufficiently large that, V_Q is to the right of the rightmost vertical chain in $G \cap R_i$, above the highest horizontal chain in $G \cap R_i$ (i.e. $Q > P + M$) and if x_i is not on the geometric boundary of R_i , the vertical chain V_{-Q} is to the left of x_i and the horizontal chain H_{-Q} is below x_i . The union of $V_{\pm Q}$ and $H_{\pm Q}$ contains an obvious ‘ring’, a connected cycle of circles, just interior to the geometric boundary of R_i . Observe that this ring meets each arc component of $\alpha_0 \cap R_i$ in two circles, which are bridges. Select a simple closed curve (in Y), S , contained in this ring, which connects with β_i at two points (one, call it y'_i , near y_i , and another, call it z'_i , near z_i), but which uses the bridges to prevent intersection with any other (arc component of) α_0 . We will modify S so that it visits x_i . If this is possible

then either the arc ‘travel along β_i from y_i to y'_i , then *clockwise* along S until we reach z'_i , followed by traveling along the arc β_i to z_i ’; or the arc obtained by following S *anti-clockwise*, is the required modification of β_i .

Two cases arise. If x_i is on the geometric boundary of R_i , we can find two arcs starting at x_i , and otherwise disjoint, both meeting S (but disjoint from the grid G). The required modification of S is now obvious (follow S , then the first arc met, to x_i , back to S along the second arc, and finish following S). If x_i is not on the geometric boundary, then it is to the left and below the grid. It is also in some rectangle, R_τ , τ from some $\Sigma_{N'}$ where $N' > N$, where R_τ is disjoint from the geometric boundary of R_i . Following S anticlockwise we can get to a point, a_i , below the lowest horizontal line of the grid, but above and to the left of the top-left corner of R_τ . Following S clockwise we can get to a point, b_i , left of the leftmost vertical line of the grid $G \cap R_i$, but below and to the right of the bottom-right corner of R_τ . We can now find disjoint arcs from a_i to the top-left corner of R_τ , and from b_i to the bottom-right corner of R_τ . And these can be extended to disjoint (except at x_i) arcs a_i to x_i and b_i to x_i . Again, using these arcs, we can modify S to detour through x_i . \square

5 Complexity

As discussed at the end of Section 2, the definition of the n -sac property uses quantification over two uncountable sets, namely all n -tuples of points in the space, and all arcs in the space. However there is a characterization of the 3-sac property in graphs which only requires quantification over countable sets. This characterization, then, is hugely simpler than the formal definition. One might hope to find similarly simple characterizations of n -sac, or ω -sac, regular curves or rational curves. In this section we show that no such simple characterization of the n -sac (or ω -sac) property exists for rational curves. Further, there is no characterization of n -sac or ω -sac general curves which does not use quantification over two uncountable sets, in other words, no characterization simpler than the definition.

More precisely we show that for $n \geq 2$ or $n = \omega$ the set of n -sac rational continua and the set of all n -sac but not $(n+1)$ -sac rational continua are Σ^1_1 -hard subsets of the space of subcontinua of \mathbb{R}^N , for $N \geq 3$. This should be interpreted as saying that there is no formula characterizing the n -sac, or ω -sac, rational curves which does not require at least one (existential) quantifier

running over an uncountable (Polish) space. We further show that, for $n \geq 2$ or $n = \omega$, the set of n -sac continua are $\mathbf{\Pi}_2^1$ -complete subsets of the space of subcontinua of \mathbb{R}^N , for $N \geq 4$. The logical interpretation of this statement is that the simplest formula characterizing the n -sac, or ω -sac, curves has exactly one universal quantifier followed by one existential quantifier running over uncountable Polish spaces — as in the formal definition of the n -sac property.

These complexity results are part of a significant body of work in descriptive set theory. In particular, Becker proved that the set of simply connected continua in \mathbb{R}^3 form a $\mathbf{\Pi}_1^1$ -complete set (see [6] 33.17), while Becker and Ajtai (independently) showed that the path connected (i.e. arc connected, or equivalently 2-sac) continua in \mathbb{R}^3 are $\mathbf{\Pi}_2^1$ -complete (see [6] 37.11). Other recent work in this area includes that of Becker and Pol [1] on arc components.

The basic notions on this section are taken from [3]. A Polish space is one which is separable and completely metrizable. For a Polish space X , denote by $\mathcal{C}(X)$ the *hyperspace of subcontinua of X* endowed with the Vietoris topology. It is Polish. A subset A of X is called *analytic* if it is the continuous image of a Polish space. The symbol $\Sigma_1^1(X)$ denotes the family of analytic subsets of X . The set of complements of elements of Σ_1^1 is denoted Π_1^1 , while the complements of continuous images of Σ_1^1 is written $\mathbf{\Pi}_1^2$.

Let X and Y be two metric spaces, and $A \subseteq X$, $B \subseteq Y$, we say that A is *Wadge reducible to B* if there is a continuous map $f : X \rightarrow Y$ such that $A = f^{-1}(B)$; and we denote this by $A \leq_W B$. Let Y be a Polish space. Let Γ be any of the families of subsets mentioned above. Then a subset B of Y is Γ -*hard* if $A \leq_W B$ for any $A \in \Gamma(X)$, where X is a zero-dimensional Polish space. If in addition $B \in \Gamma(Y)$, then we say that B is Γ -*complete*. The point here is that, for example, any Σ_1^1 -hard set is at least as complex as any analytic set, and any formula describing it must contain at least one existential quantifier running over an uncountable Polish space. Further, if a Γ -hard set, A say, Wadge reduces to another set B , then B is also Γ -hard. This gives a standard method of proving that a given set of interest is, say, a Σ_1^1 -hard set — show that a known Σ_1^1 -hard set reduces onto it.

Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences on \mathbb{N} , including the empty sequence, $()$. Given $s = (s_1, s_2, \dots, s_n)$ and k , let $s \frown k = (s_1, \dots, s_n, k)$. A *tree on \mathbb{N}* is a subset τ of $\mathbb{N}^{<\mathbb{N}}$ which is closed under initial segments, in other words if $t \in \tau$ and $s = t \upharpoonright m$ for some $m \leq \text{length}(t)$, then $s \in \tau$. Identifying a subset of $\mathbb{N}^{<\mathbb{N}}$ with its characteristic function, let \mathbf{Tr} be the subspace of $\{0, 1\}^{\mathbb{N}^{<\mathbb{N}}}$ of all trees on \mathbb{N} . It is a closed subset, hence compact. A tree with

an infinite branch is said to be *ill-founded*. Denote by \mathbf{IF} the space of all ill-founded trees on \mathbb{N} . It is well known that \mathbf{IF} is Σ_1^1 -complete, see Theorem 27.1 and page 240 of [6].

Now our approach to showing that, for $n \geq 2$ or $n = \omega$ the set of n -sac rational continua and the set of all n -sac but not $(n+1)$ -sac rational continua are Σ_1^1 -hard sets, is clear — we will find continuous reductions from \mathbf{IF} onto these sets. This entails constructing, for a given tree, a suitable rational continua. The next few lemmas provide building blocks (‘tiles’) and tools for making complex rational continua.

A *tile* is any space T which is (i) a subspace of the solid square pyramid in \mathbb{R}^3 with base $S = [-1, +1]^2 \times \{0\}$ and vertex at $(0, 0, 1)$ (so it has height 1) and (ii) contains the four corner points of the base, (i, j) for $i, j = \pm 1$. Call the intersection of a tile T with S , the *base of T* . Call the intersection of T with the *boundary* $B = ([-1, 1] \times \{-1, 1\} \times \{0\}) \cup (\{-1, 1\} \times [-1, 1] \times \{0\})$ of the base S , the *boundary of T* . Call the point $(-1, 1, 0)$ the *top-left corner of the base*.

Lemma 12. *There are (homeomorphic) tiles T_0 and T_1 such that: (i) T_0 and T_1 are ω -sac rational curves, (ii) the boundary of T_0 is B , and (iii) the boundary of T_1 is $(A \times \{-1, 1\} \times \{0\}) \cup (\{-1, 1\} \times A \times \{0\})$ where $A = \{-1, 0, 1\} \cup \{-2^{-n} : n \in \mathbb{N}\}$.*

Proof. The example, Y , of an ω -sac rational curve given in Theorem 11 is derived from a space X . This space X is a subspace of $[-1, +1]^2$. We may suppose that X is in fact a subspace of the square $S = [-1, +1]^2 \times \{0\}$. The space Y is obtained from X by identifying a sequence of pairs of double sequences. These double sequences all are disjoint from the boundary, B , of the square S , and the diameters and distance between pairs of sequences converges to zero. This identification process can be repeated in $(-1, +1)^2 \times \mathbb{R}$, keeping the boundary, B , of the square, S , fixed, to get a space T'_0 homeomorphic to Y . Applying a homeomorphism of $[-1, +1]^2 \times \mathbb{R}$ fixing B , and changing only the z -coordinates, to T'_0 , we get a space T_0 , also homeomorphic to Y and containing B , and which is contained in the pyramid with base $[-1, +1]^2 \times \{0\}$ and height 1. Thus T_0 is a tile.

Scaling \mathbb{R}^3 around the center point of the the base square, S , we can shrink T_0 away from the boundary B of S and still have it inside the required pyramid. Instead of doing this transformation, shrink T_0 while keeping fixed the set $(A \times \{-1, 1\} \times \{0\}) \cup (\{-1, 1\} \times A \times \{0\})$. This gives T_1 . \square

Let X be a space and A an infinite subset. We say that X is $\omega\text{-sac}^+$ (with respect to A) if for any points x_1, \dots, x_n in X there is an arc α in X visiting the x_i in order, such that α meets A only in a finite set. Observe that if X is $\omega\text{-sac}^+$ with respect to A , and A' is an infinite subset of A , then X is $\omega\text{-sac}^+$ with respect to A' .

Lemma 13 (ω -Gluing). *Let $Z = X \cup Y$, where X, Y and $A = X \cap Y$ are infinite. If X is $\omega\text{-sac}^+$ with respect to A , and Y is $\omega\text{-sac}$, then Z is $\omega\text{-sac}$.*

Proof. Take any finite sequence of points z_1, \dots, z_N in Z . By adding points to the start and end of the sequence, if necessary, we can suppose that z_0 and z_N are in X . Group the sequence, $z_1, \dots, z_{n_1}, z_{n_1+1}, \dots, z_{n_2}, \dots, z_{n_{k-1}}, z_{n_{k-1}+1}, \dots, z_{n_k}$, where z_1, \dots, z_{n_1} are in X , $z_{n_1+1}, \dots, z_{n_2}$ are in $Y \setminus X$, and so on, until $z_{n_{k-1}+1}, \dots, z_{n_k} = z_N$ are in X . Pick t_1^\pm, \dots, t_k^\pm in $A \setminus \{z_i\}_{i \leq N}$.

Using the fact that X is $\omega\text{-sac}^+$, pick arc α^- in X visiting in order, $z_1, \dots, z_{n_1}, t_1^-, t_1^+, z_{n_2+1}, \dots, z_{n_2+1}, \dots, z_{n_3}, t_2^-, t_2^+$ and so on, ending with z_{n_k} , such that α^- meets A only in a finite set F .

Using the fact that Y is $\omega\text{-sac}$, pick an arc α^+ in Y visiting in order the points, $t_1^-, z_{n_1+1}, \dots, z_{n_2}, t_1^+, t_2^-$ and so on, avoiding $F \setminus \{t_1^\pm, \dots, t_k^\pm\}$.

Now we can interleave α^- and α^+ to get an arc, α , visiting all the specified points in order. So we start α by following α^- to visit z_1, \dots, t_1^- , then pick up α^+ at t_1^- to visit $z_{n_1+1}, \dots, z_{n_2}, t_1^+$, and back to α^- from t_1^+ , and so on. \square

Lemma 14.

- (i) *The tile T_0 is $\omega\text{-sac}^+$ with respect to any infinite discrete subset of its boundary.*
- (ii) *The tile T_1 is $\omega\text{-sac}^+$ with respect to its boundary.*

Proof. Recall that T_0 and T_1 are both homeomorphic. In turn, T_0 is a homeomorph of Y from Theorem 11 with the boundary square for both not just homeomorphic but identical (when we identify the plane, \mathbb{R}^2 , with $\mathbb{R}^2 \times \{0\}$). So we argue this for Y only. Looking at the proof that Y is $\omega\text{-sac}$ it is clear that the arc, α_0 , visiting some specified points, x_1, \dots, x_n , in order, need only touch the boundary in an arbitrarily small neighborhood of any x_i which happens to be on the boundary. This immediately gives the first claim — Y (and so T_0) is $\omega\text{-sac}^+$ with respect to infinite discrete subsets of the boundary square.

Further, the point $(0, -1)$ can be reached from the interior of Y (away from the boundary square) by two disjoint arcs which meet the set $(A \times \{-1, 1\}) \cup$

$(\{-1, 1\} \times A)$ only at $(0, -1)$ — for one arc, α^- , follow one side of the sequence of circles converging to $(0, -1)$ and for the other, α^+ , start at $(0, -1)$ go right along the boundary edge a short way, and then go into the interior. The same is true for the points $(0, 1)$, $(-1, 0)$, and $(1, 0)$.

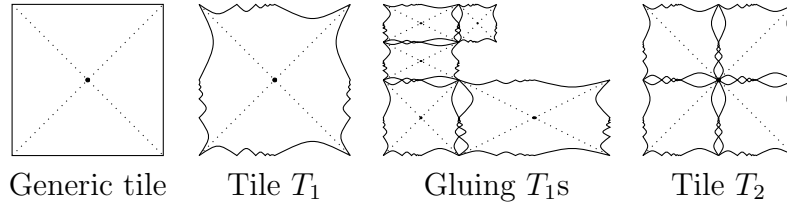
Now to get the desired arc, if every x_i is not one of $(0, -1)$, $(0, 1)$, $(-1, 0)$, or $(1, 0)$, then just use α_0 . While if x_i is say, $(0, -1)$, then pick α_0 to visit $x_1, \dots, x_{i-1}, t^-, t^+, x_{i+1}, \dots$, where t^-, t^+ are points close to $(0, -1)$ on α^- and α^+ respectively. Now let α be the arc that follows α_0 to t^- , then follows α^- to $x_i = (0, -1)$, then α^+ to t^+ , and then resumes along α_0 . \square

For any tile T , $\mathbf{x} = (x, y)$ in \mathbb{R}^2 and $a, b > 0$, denote by $T(\mathbf{x}, a, b)$ the space T scaled in the x and y coordinates so its base has length a and width b , then scaled in the z coordinate so that the pyramid containing it has height no more than the smaller of a and b , and then translated in the x, y -plane so that the top-left corner is at $(x, y, 0)$.

From Lemma 13, part (ii) of Lemma 14, and an easy induction argument, the following is clear.

Lemma 15. *Any space obtained by gluing along matching edges a finite family of translated and scaled copies of T_1 is a rational ω -sac curve.*

We define recursively a sequence of tiles. The first in the sequence is T_1 from above. Given tile T_n , where $n \geq 1$, define T_{n+1} to be $T_n((-1, 1), 1, 1) \cup T_n((-1, 0), 1, 1) \cup T_n((0, 1), 1, 1) \cup T_n((0, 0), 1, 1)$ scaled in the z -coordinate only so as to fit inside the pyramid with base S and height 1. Then all the tiles T_n are rational ω -sac continua.



Theorem 16. *Fix $N \geq 3$. For $n \geq 2$ or $n = \omega$, let R_n be the set of rational n -sac continua, and let $R_{n, \neg(n+1)}$ be the set of rational continua which are n -sac but not $(n+1)$ -sac.*

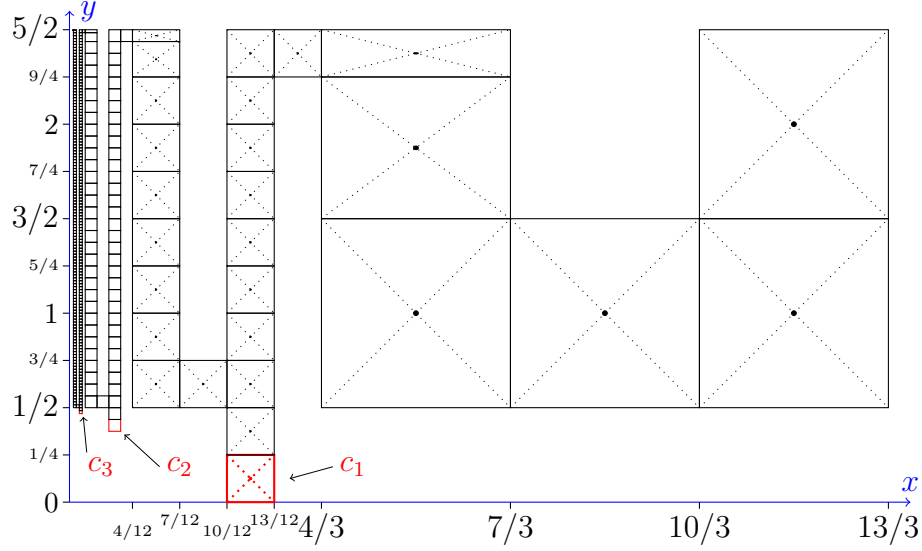
Then all the sets R_n and $R_{n, \neg(n+1)}$ are Σ_1^1 -hard subsets of the space $\mathcal{C}(\mathbb{R}^N)$.

Proof. We prove that there is a continuous map K of the space \mathbf{Tr} of all trees on \mathbb{N} into the space $\mathcal{C}(\mathbb{R}^3)$ such that: if the tree τ has no infinite branch then K_τ is a rational continuum which is not arc-connected (in other words, 2-sac), while if τ has an infinite branch, then K_τ is an ω -sac rational continuum. The claim that R_n is a Σ_1^1 -hard subset of the space $\mathcal{C}(\mathbb{R}^N)$, follows simultaneously for all n and N . We then give the minor modifications necessary to have that K_τ is n -sac but not $(n+1)$ -sac when τ has an infinite branch. The remaining claims follows immediately.

A basic building block for K_τ is $S(T)$ a variant of the topologist's sine-curve based on a tile T . This sine-curve lies in the rectangular box $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 13/3, 0 \leq y \leq 5/2, 0 \leq z \leq 1\}$. We call the point $(0, 5/2, 0)$ the top left corner of $S(T)$.

Explicitly $S(T)$ is $(D_0 \cup AB_0 \cup U_0 \cup AT_0) \cup \bigcup_{n \geq 1} (D_n \cup C_n \cup AB_n \cup U_n \cup AT_n)$ where

$$\begin{aligned}
D_n &= \bigcup_{i=1}^{2 \cdot 4^n} T(10/(3 \cdot 4^n), 1/2 + i/4^n, 1/4^n), \\
C_n &= T(10/(3 \cdot 4^n), 1/2 - 1/4^n, 1/4^n) \cup T(10/(3 \cdot 4^n), 1/2, 1/4^n) \quad (n \geq 1), \\
AB_n &= T((7/(3 \cdot 4^n), 1/2 + 1/4^n), 1/4^n), \\
U_n &= \bigcup_{i=1}^{2 \cdot 4^n - 1} T((4/(3 \cdot 4^n), 1/2 + i/4^n), 1/4^n) \\
&\quad \cup T((4/(3 \cdot 4^n), 5/2 - 1/4^{n+1}, 1/4^n, 3/4^{n+1}) \\
&\quad \cup T((4/(3 \cdot 4^n), 5/2, 1/4^n, 1/4^{n+1}), \quad \text{and} \\
AT_n &= T((4/(3 \cdot 4^n) - 1/4^{n+1}, 5/2, 1/4^{n+1}, 1/4^{n+1}).
\end{aligned}$$



For each $m \geq 1$, let $c_m = T(10/(3 \cdot 4^n), 1/2 - 1/4^n, 1/4^n)$ be the tile in $S(T)$ at the bottom of the m th connector, C_m .

For any point y in \mathbb{R} , and any $a > 0$, let $S(y, a, T)$ be the sine curve $S(T)$ scaled (in all directions) by a , and translated so that its top left corner is at $(0, y, 0)$.

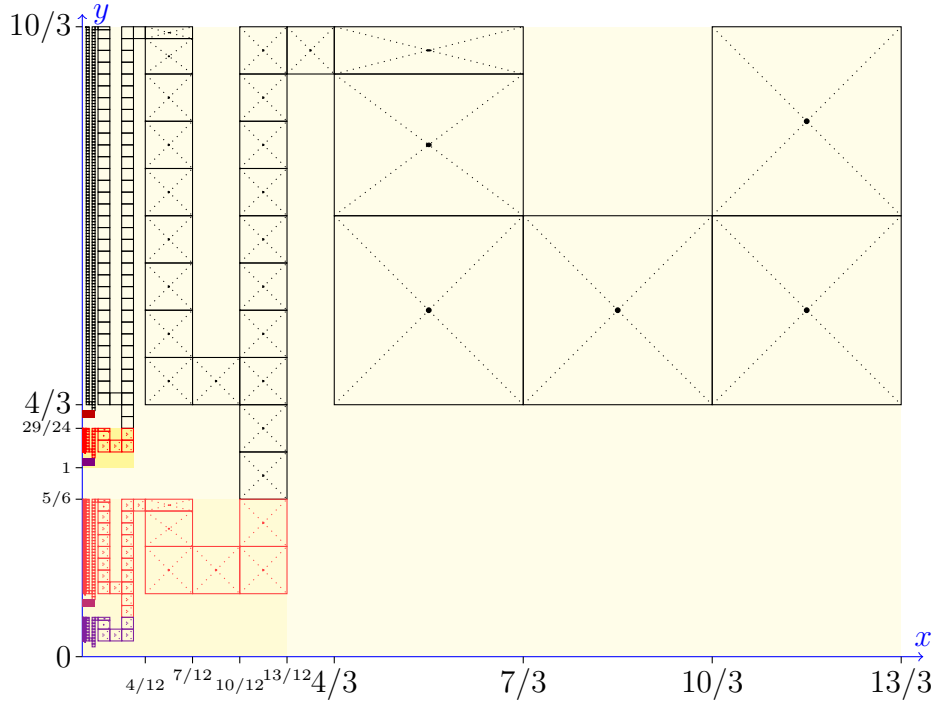
Next, given a tree τ and a tile T , we define a ‘branch space’, $B(T, \tau)$, lying in the rectangular box, $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 13/3, 0 \leq y \leq 10/3, 0 \leq z \leq 1\}$, which is $\bigcup \{S_s : s \in \tau\}$, where each S_s is defined with the aid of some connecting tiles, c_s , and numbers, y_s , by induction on the height of s , as follows.

Step 1 Let $y_{()} = 5/2 + 5/6 = 10/3$, and let $S_{()} = S(y_{()}, 1, T)$ (i.e. the sine-curve defined above based on T , translated along the y -axis by $5/6$).

Step 2 The sine curve, $S_{()}$ has a family of connecting tiles c_m . Set $c_{(m)} = c_m$. Let $y_{(m)} = y_{()} - 2/4^0 - 2/4^m$, and let $S_{(m)} = S(y_{(m)}, \frac{1}{4^m}, T)$. Note, critically, that the top-right tile of this sine-curve, $S_{(m)}$, is such that its top edge coincides with the bottom edge of $c_{(m)}$.

Step $n + 1$ Fix an $s \in \tau$ with length n . We again will have connecting tiles, c_m , from the sine-curve S_s . Set $c_{s \frown m} = c_m$. Let $y_{s \frown m} = y_s - 2/4^L - 2/4^{L+m}$ where $L = \sum_{i=1}^n s_i$, and let $S_{s \frown m} = S(y_{s \frown m}, 1/4^{L+m}, T)$. Again note that the top-right tile of $S_{s \frown m}$ has its top edge coinciding with the bottom edge of $c_{s \frown m}$.

Assume, for this paragraph only, that $\tau = \tau_c$ is the complete tree, and T is the solid tile. For any s in τ , let $\tau_s = \{s' \in \tau : s' \text{ extends } s\}$, and let $B_s = \bigcup \{S_{s'} : s' \in \tau_s\}$. By construction, $B_{(1)}$ is a $1/4$ th copy of $B(T, \tau) = B_{(0)}$, and $B_{(2)}$ is a $1/16$ th copy. It is easy to check that the height (in the y -coordinate) of $B_{(0)}$ is exactly $10/3$. So the height of $B_{(2)}$ is $1/16$ th of this, which is $5/24$. The gap between the top edge of $B_{(1)}$ and the top edge of $B_{(2)}$ is $9/24$. Thus $B_{(2)}$ is disjoint from $B_{(1)}$. By self-similarity it follows that B_s and B_t meet if and only if one of s and t is an immediate successor of the other. This all shows that, for any tree and any tile, $B(T, \tau)$ is well defined, and is the edge connected union of tiles meeting along matching edges.



We call the point $(0, 10/3, 0)$ the top left corner of $B(T, \tau)$. For y in \mathbb{R} and $a > 0$, let $B(y, a, T, \tau)$ be $B(T, \tau)$ scaled in the y -coordinate only by a , and translated so its top left corner is at $(0, y, 0)$.

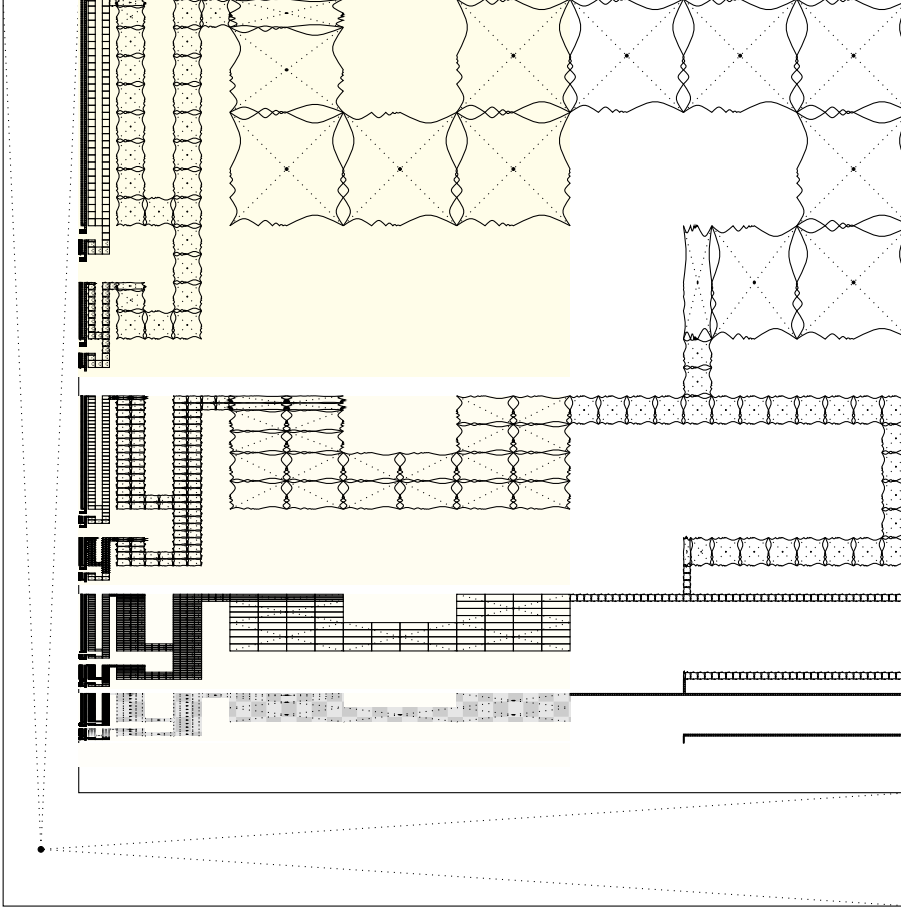
Now our K_τ will consist of $\bigcup_{n \geq 0} B_n \cup L \cup S$, where $B_n = B(y_n, 1/2^n, T_{n+1}, \tau)$, for $y_n = 7/2^n$, and the two pieces L and S are defined as follows.

The set L is a homeomorphic copy of the tile T_0 , bent in the middle so that its base is contained in the L -shaped area $\{(x, y, 0) \in \mathbb{R}^3 : -2/3 \leq x \leq$

$0, -1 \leq y \leq 7$ or $0 \leq x \leq \frac{22}{3}, -1 \leq y \leq 0\}$ and the boundary of the base of the tile is the boundary of this area.

The set S is a sine curve variant based on the tile T_1 , which connects the branch spaces B_n , and converges down to the x -axis. Concretely, $S = \bigcup_{n \geq 0} (AR_n \cup D_n \cup AL_n \cup C_n)$ where

$$\begin{aligned}
AR_n &= \bigcup_{i=0}^{3 \cdot 4^n - 1} T_1((13/3 + i/4^n, 7/2^n), 1/4^n), \\
D_n &= \bigcup_{i=1}^{3 \cdot 2^n - 1} T_1((13/3 + 3 - 1/4^n, 7/2^n - i/4^n), 1/4^n), \\
AL_n &= \bigcup_{i=1}^{2 \cdot 4^n - 2} T_1((13/3 + 1 + i/4^n, 7/2^n + 1/4^n - 3/2^n), 1/4^n, 1/4^n) \\
&\quad \cup T_1((13/3 + 1, 7/2^n + 1/4^n - 3/2^n), 1/4^{n+1}, 1/4^n) \\
&\quad \cup T_1((13/3 + 1 + 1/4^n, 7/2^n + 1/4^n - 3/2^n), 3/4^{n+1}, 1/4^n), \quad \text{and} \\
C_n &= \bigcup_{i=1}^{2^{n+1}} T_1(13/3 + 1, 7/2^{n+1} + i/4^{n+1}, 1/4^{n+1}).
\end{aligned}$$



Claim 1. K_τ is a rational continuum.

Proof: Let $R = \bigcup_n B_n \cup S$. Let $L' = \{(x, y, 0) \in \mathbb{R}^3 : x = 0, -1 \leq y \leq 7 \text{ or } -2/3 \leq x \leq 22/3, y = 0\}$, be the inner boundary of the base of L .

Since $\overline{R} \subseteq R \cup L'$, K_τ is clearly compact. Since L and R are connected, and S is a variant topologists sine curve, clearly K_τ is connected.

For all the points of K_τ except those on L' , we have a natural neighborhood base at the point for which each element has a countable boundary (which comes from the tile(s) the point is in).

Take any point \mathbf{x} in L' . We suppose now, $\mathbf{x} = (x_0, 0, 0)$ (the other case is similar). Because B_n is based on the tile T_n , combined with the fact that the T_1 s in the connecting sine curve, S , have size shrinking to zero, the set M of all x -components of the left and right edges of the base of tiles in R is dense in $[0, 22/3]$.

Let U be a rectangular neighborhood of \mathbf{x} in \mathbb{R}^3 , and $r_{min} = \min\{x : (x, 0, 0) \in U\}$ and $r_{max} = \max\{x : (x, 0, 0) \in U\}$. Without loss of generality, if y_{max} is the value of the maximum y -component in U then $\{(x, y, z) \in U \mid y = y_{max}\}$ do not intersect with any of the B_n , i.e. the top of U is in between B_n and B_{n+1} for some n .

The set $U \cap L$ includes a neighborhood N of x which has countable boundary. Let $a = \min\{x : (x, 0, 0) \in N\}$ and $b = \max\{x : (x, 0, 0) \in N\}$. Since M is dense there are sequences $(a_n)_n$ and $(b_n)_n$ in M such that a_n increases to a , b_n decreases to b , and for each n , $r_{min} \leq a_n \leq a < b \leq b_n \leq r_{max}$. Let $m_1 \geq n$ be such that both of the lines $x = a_1$ and $x = b_1$ intersect the xy -projection of $B_{m_1} \cup \{(x, y, z) \in S : y \leq 7/2^{m_1}\}$ along edges of tiles only. Let $S_n = \{(x, y, z) \in S : y \leq 7/2^n\}$. And inductively, let $m_i \geq m_{i-1}$ such that the lines $x = a_i$ and $x = b_i$ intersect the xy -projection of $B_{m_i} \cup S_{m_i}$ along edges of tiles only. Now take $N' = \bigcup_i ((S_{m_i} \setminus S_{m_{i+1}} \cup \bigcup_{k+m_i < m_{i+1}} B_{m_i+k}) \cap \{(x, y, z) : a_i \leq x \leq b_i\})$.

Here for each i , we cut B_{m_i+k} along edges of finitely many tiles, hence the boundary is at most countable. And similarly for $S_{m_i} \setminus S_{m_{i+1}}$, we cut along the edges of finitely many tiles. Thus N' has countable boundary. Moreover, $N \cup N' \subset R$ is a neighborhood of \mathbf{x} with countable boundary. \blacksquare

Claim 2. If τ has an infinite branch then K_τ is ω -sac.

Proof: Suppose τ has an infinite branch. Note that if T is any tile, then there is a branch of edge connected tiles in $B(T, \tau)$ which converges to a point \mathbf{y}_τ on the y -axis.

We first show that for any $m \geq 1$, the branch space $B(T_m, \tau) \cup \{\mathbf{y}_\tau\}$ is ω -sac. To do so we only need to check that if \mathbf{y}_τ is one of the n -points x_1, \dots, x_n in $B(T_m, \tau) \cup \{\mathbf{y}_\tau\}$, then we can find an arc joining them in that order. Suppose $x_k = \mathbf{y}_\tau$. Then the points $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ are in some finite family of edge connected tiles of $B(T_m, \tau)$. Let t be a tile in the branch space $B(T_m, \tau)$ such that none of the x_i s is in the tiles to the left and bottom of this tile except for $x_k = \mathbf{y}_\tau$. Let \mathbf{y}_1 be the bottom left corner of t , and \mathbf{y}_2 be the top left corner of t . Then by Lemma 15 there is an arc α_0 in $B(T_m, \tau)$ through the points $x_1, \dots, x_{k-1}, \mathbf{y}_1, \mathbf{y}_2, x_{k+1}, \dots, x_n$ in the given order. Let α_1 be the part of α_0 through $x_1, \dots, x_{k-1}, \mathbf{y}_1$. Let β_1 be the arc starting at \mathbf{y}_1 and ending at \mathbf{y}_τ obtained by traveling along the right and bottom edges of tiles of the branch converging to \mathbf{y}_τ . Similarly, let α_2 be the part of α_0 through $\mathbf{y}_2, x_{k+1}, \dots, x_n$. And let β_2 be the arc starting at \mathbf{y}_τ , and

following the left and top edges of tiles of the branch converging to \mathbf{y}_τ , back to \mathbf{y}_2 . Then the arc α obtained by following $\alpha_1, \beta_1, \beta_2$ and then α_2 , is the desired arc through the points x_1, \dots, x_n in the given order.

Back now to K_τ , when τ has an infinite branch. For each n , B_n has a corresponding branch converging to a point \mathbf{y}_n on the y -axis. An easy modification of the argument for $B(T_m, \tau) \cup \{\mathbf{y}_\tau\}$ shows that the space $R \cup \{\mathbf{y}_n : n \in \mathbb{N}\}$ is ω -sac.

Since $R \cup \{\mathbf{y}_n\}_n$ is ω -sac, $(R \cup \{\mathbf{y}_n\}_n) \cap L = \{\mathbf{y}_n\}_n$, and L is ω -sac⁺ with respect to discrete sets (Lemma 14, part (i)), it follows from the ω -Gluing Lemma that $K_\tau = R \cup L$ is indeed ω -sac. \blacksquare

Claim 3. If τ has no infinite branch then K_τ is not 2-sac.

Proof: If τ does not have any infinite branches, then there are no arcs connecting L to R . This is clear because, without infinite branches, any path starting in R and attempting to reach L is forced to travel along a topologist's sine curve variant — which is impossible. \blacksquare

Claim 4. The map $\tau \mapsto K_\tau$ is continuous.

Proof: Let $K : \mathbf{Tr} \rightarrow \mathcal{C}(\mathbb{R}^3)$ given by $K(\tau) = K_\tau$. Let s be in $\mathbb{N}^{<\mathbb{N}}$, and write $[s]$ for the set of all trees containing s . Then $[s]$ is a closed and open subset of \mathbf{Tr} . Subbasic open sets in $\mathcal{C}(\mathbb{R}^3)$ are of one of two forms: (i) $\langle U \rangle = \{C : C \subseteq U\}$ and (ii) $\langle X; V \rangle = \{C : C \cap V \neq \emptyset\}$, where U and V are open subsets of \mathbb{R}^3 . We show inverse images under K of both types of subbasic open set are open in \mathbf{Tr} , thus confirming continuity of the map $\tau \mapsto K_\tau$.

For subbasic sets of type (ii), the sets V may be taken to come from any basis for \mathbb{R}^3 ; we will take for V open balls in \mathbb{R}^3 which either meet, or have closure disjoint from, $L \cup S$. Fix such a V . If V meets $L \cup S$, then $K^{-1}\langle X; V \rangle = \mathbf{Tr}$. If the closure of V is disjoint from L , then for any tree τ , V meets only finitely many $B_n(\tau)$, and in each of these branch spaces, meets only finitely many sine curves. Suppose V meets sine curves labeled by s_1, \dots, s_k . Then $K^{-1}\langle X; V \rangle = \bigcup \{[s_i] : 1 \leq i \leq k\}$, which is open (each $[s_i]$ is open).

For subbasic sets of type (i), if $L \cup S$ is not contained in U , then $K^{-1}\langle U \rangle = \emptyset$. So suppose, $L \cup S \subseteq U$. Let τ_c be the complete tree. Then all but finitely many of the sine curves making up the $B_n(\tau_c)$ s are contained in U . Let them

be labeled by s_1, \dots, s_k . Then $K^{-1}\langle U \rangle = \mathcal{C}(\mathbb{R}^3) \setminus \bigcup \{[s_i] : 1 \leq i \leq k\}$, which is open (each $[s_i]$ is closed). \blacksquare

Claims 1–4 show that $\tau \mapsto K_\tau$ is as required — continuous, and K_τ is a rational continuum which is ω -sac if τ has an infinite branch, but is not even 2-sac when τ has no infinite branches.

We now turn to the case for $R_{n, \neg(n+1)}$. To start fix $n \geq 2$. Select $n - 2$ points a_1, \dots, a_{n-2} from the interior of the right hand edge of the base of T_0 . Similarly to the definition of T_1 , shrink T_0 while keeping fixed the set $\{a_1, \dots, a_{n-2}\}$ and the top edge of the base. This gives a tile \hat{T}_n . Now consider the map $\tau \mapsto K_\tau^n$ where K_τ^n is K_τ along with the tile $\hat{T}_n(22/3, 0, 1)$. Then it is easy to see (given our previous work) that K_τ^n is a rational continuum and the map $\tau \mapsto K_\tau^n$ is continuous. Because the extra tile, $\hat{T}_n(22/3, 0, 1)$, meets the rest of K_τ^n in exactly $n - 1$ points (namely a_1, \dots, a_{n-2} and the topleft corner of the base of the tile), K_τ^n is never $(n + 1)$ -sac. When τ has no infinite branch, then K_τ^n is not 2-sac, so definitely not in $R_{n, \neg(n+1)}$. But when τ has an infinite branch, both $\hat{T}_n(22/3, 0, 1)$ and the rest of K_τ^n are ω -sac, and (again) meet in $n - 1$ points — so by Lemma 9 is n -sac. \square

By definition of ω -sac, the set of all ω -sac curves is a $\mathbf{\Pi}_2^1$ set. It turns out there is no simpler characterization for being ω -sac. Given the previous theorem, the proof of the following result is very similar to that of Becker and (independently) Ajtai that the set of arc connected subcontinua of \mathbb{R}^3 is $\mathbf{\Pi}_2^1$ -complete (see [6] 37.11). Consequently we give just a sketch of the proof, highlighting the differences.

Theorem 17. *The sets S_n of n -sac curves, for all $n \geq 2$ and $n = \omega$, are $\mathbf{\Pi}_2^1$ -complete subsets of the space $\mathcal{C}(\mathbb{R}^N)$, where $N \geq 4$.*

Proof. We prove the claim, for all n and N simultaneously, by proving that there is a continuous map Φ from the space $\mathbb{N}^\mathbb{N}$ into the space $\mathcal{C}(\mathbb{R}^4)$ such that: given a $\mathbf{\Pi}_2^1$ set $A \subset \mathbb{N}^\mathbb{N}$, if $x \in A$ then $\Phi(x) = P_x$ is a curve which is not arc connected (i.e. 2-sac), while if $x \in A$, then P_x is an ω -sac curve.

Let A be a $\mathbf{\Pi}_2^1$ subset of $\mathbb{N}^\mathbb{N}$ and B be a $\mathbf{\Sigma}_1^1$ subset of $\mathbb{N}^\mathbb{N} \times 2^\mathbb{N}$ with $x \in A$ if and only if for each $y \in 2^\mathbb{N}$, $(x, y) \in B$. Now let τ be a tree on $\mathbb{N} \times 2 \times \mathbb{N}$ with $B = \{(x, y) : \exists z \in \mathbb{N}^\mathbb{N}(x, y, z) \text{ is a branch of } \tau\} = \{(x, y) : \tau(x, y) \in \mathbf{IF}\}$. Recall that $\tau(x, y) = \{s : (x \upharpoonright \text{length}(s), y \upharpoonright \text{length}(s), s) \in \tau\}$ is a tree on \mathbb{N} .

Now for each $x \in \mathbb{N}^{\mathbb{N}}$, we will construct a curve $P_x \subset \mathbb{R}^4$ as follows. First, we identify the Cantor space $2^{\mathbb{N}} \subset \mathbb{N}^{\mathbb{N}}$ with the standard Cantor set in $[0, 1]$. Then for each $y \in 2^{\mathbb{N}}$, let $L_{x,y} = K_{\tau(x,y)}$, as described in Theorem 16, placed in the cube $\{(a, b, c, d) : -\frac{2}{3} \leq a \leq \frac{22}{3}, -1 \leq b \leq 7, c \geq 0, d = y\}$. Thus the outside edges of the tile L in $K_{\tau(x,y)}$ is on $a = -\frac{2}{3}$ or $b = -1$. Let $P_x = \bigcup_{y \in 2^{\mathbb{N}}} L_{x,y} \cup M$. Here we have connected the continua $L_{x,y}$ along the edges on $a = -\frac{2}{3}$ by adding the curve M , which is a copy of the Menger cube scaled and translated inside the cube $\{(a, b, c, d) : -2 \leq a \leq -\frac{2}{3}, -1 \leq b \leq 7, c = 0, 0 \leq d \leq 1\}$.

Then, P_x is a curve and the map $x \mapsto P_x$ from $\mathbb{N}^{\mathbb{N}} \rightarrow \mathcal{C}(\mathbb{R}^4)$ is continuous. Moreover, $x \in A$ if and only if for each $y \in 2^{\mathbb{N}}$, $\tau(x, y) \in \mathbf{IF}$. Thus if $x \notin A$, then there is $y \in 2^{\mathbb{N}}$ with $\tau(x, y) \notin \mathbf{IF}$, so the corresponding rational ω -sac continua $L_{x,y} = K_{\tau(x,y)}$ is not 2-sac, hence the union $P_x = M \cup \bigcup_{y \in 2^{\mathbb{N}}} L_{x,y}$ is also not 2-sac.

On the other hand, if $x \in A$, then for each $y \in 2^{\mathbb{N}}$, $L_{x,y}$ is ω -sac by Theorem 16. It is straightforward to check that the Menger cube M is ω -sac. The proof that P_x is ω -sac is now straightforward given the techniques developed for Theorem 16. \square

Since the set of ω -sac curves is Π_2^1 -complete it is not a Σ_1^1 -set (i.e. an analytic set). From which the next corollary easily follows.

Corollary 18. *There is no mapping universal for the class of all ω -sac curves. More generally, given any countable family of ω -sac curves, K_n for n in \mathbb{N} , there is an ω -sac curve L which is not the continuous image of any K_n .*

Open Problems

Our results raise a number of questions about optimality. For a space X , define the *sac number* of X , denoted $\text{sac}(X)$, to be the maximal n such that X is n -sac, or ∞ if X is ω -sac.

- What is the sac number of the N -trix? We know $\text{sac}(3\text{-trix}) = \text{sac}(4\text{-trix}) = 3$ but $\text{sac}(5\text{-trix}) = 5$, and in general $\text{sac}(N\text{-trix}) \leq N$.
- What is the sac number of a closed disk with N -handles? What is the sac number of (compact) 2-manifolds? What is the sac number of finite simplicial 2-complexes?

We have shown that the 3-sac graphs can be simply characterized, but (for $n \geq 2$ or $n = \omega$) there is no simple characterization of n -sac rational continua.

- Characterize (simply) the regular n -sac curves, or prove that no such characterization is possible.

The n -sac property is a very natural strengthening of arc connectedness. Whenever a space is known to be arc connected (i.e. 2-sac) we are lead to ask for which n it is n -sac. For example the hyperspaces 2^X and $\mathcal{C}(X)$ of compact subsets and subcontinua of a continuum X are well known to be arc connected.

- When is 2^X or $\mathcal{C}(X)$ n -sac or ω -sac? We know that for hereditarily indecomposable X the space of subcontinua is not 3-sac.
- What about when we restrict to, say, locally connected continua?
- Are there, for each n , continua which are n -sac but not $(n + 1)$ -sac?

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